Making Rhythmic Canons

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Abstract

A canon is a polyphonic piece whose voices lead the same melody with different delays. A rhythmic canon is the one whose tone onsets result in a regular pulse train with no simultaneous tone onsets at a time.

Tiling in mathematics is covering an area (e.g., a square) by disjoint equal £gures. In that sense, a rhythmic canon tiles the time, providing a covering of a regular pulse train by disjoint equal rhythmic patterns.

We consider the general case of tiling the time by rhythmic patterns of a few shapes. In particular, we construct a rhythmic canon from a theme, its augmentation (the theme with all durations doubled) and its double augmentation. For this purpose we develop a tiling algorithm which resembles the sieve of Eratosthene (284–192 BC) for £nding prime numbers. Besides, we approach to an analytical solution by reformulating the problem in terms of polynomials and reducing it to Diophantine equations (supposedly 325–409). Finally, we describe the application of the methods developed to making a composition "Eine kleine Mathmusik".

1 Introduction

Recently a number of advanced mathematical models for music analysis and composition have appeared (e.g., Mazzola 1990). One of such models is due to Vuza (1991–93, 1995) who has developed a model for £nding some particular rhythms. The problem formulation has been in¤uenced by Vieru (1993).

The Vieru's and Vuza's goal was to generalize Messian's (1944) ideas backing his *modes of limited transposition*. Recall that Messian considered disjoint pitch classes with the same interval relations (= transpositions of one pitch class) which cover the 12-tone tempered scale. For instance, pitch class $\{c, e\flat, f\sharp, a\}$ and its two transpositions, by one and by two semitones, meet this requirement. This is similar to what is called in mathematics *tiling*, that is, covering an area (e.g., a square) by disjoint equal £gures.

Instead of the tempered scale, Vieru und Vuza have considered a regular pulse train. By analogy with covering the scale by a pitch class and its transpositions, the pulse train had to be covered by a certain rhythmic pattern with different delays. The disjointedness of pitch classes implied no common beats in different instances of the rhythmic pattern. The circularity of pitch (= octave periodicity) corresponded to the circular time (= periodicity of rhythms).

Such 'rhythms of limited transposition' were intended for constructing unending (= in£nite, periodic) canons. Recall that a *canon* is a polyphonic piece whose voices lead the same melody with different delays. A *rhythmic canon* is the one whose tone onsets result in a regular pulse train with no simultaneous tone onsets at a time. In that sense, a rhythmic canon tiles the time, covering a regular pulse train by disjoint equal rhythms from different voices.

From the musical standpoint, the time-tiling approach supports building polyphonic pieces from a single rhythmic pattern. It meets the principle of economy in both classical and contemporary music (long phrases built from the opening four-note motive in Beethoven's Fifth Symphony, 12-tone composition, etc.). On the other hand, in rhythmic canons the independence of voices is maximal, since no two tones occur simultaneously. Finally, rhythmic canons are harmonically unstable due to asynchronous anticipations, suspensions, and resolutions in different voices, which is much appreciated in polyphony, where cadences are usually avoided.

Therefore, it is not surprising that the time-tiling approach has attracted attention of music theorists (Andreatta et al. 2001, Fripertinger 2002). However it turned out that solutions to the time-tiling problem are mainly trivial and musically not interesting. A typical solution is a metronome rhythm entering with equal delays, e.g., a sequence of every fourth beat, entering at the £rst, at the second, and at the third beat (a rhythm analogy of the transpositions of pitch class $\{c, e\flat, f\sharp, a\}$). Non-trivial solutions have been found by Vuza for a circular time with periods $72, 108, 120, \ldots$, meeting some factorization requirements.

As one can imagine, these solutions are rather complex to make perceptible musical structures, so that the effect is as described by Xenakis (1971, p.8): "Linear polyphony destroys itself by its very complexity; what one hears is in reality nothing but a mass of notes in various registers . . . There is consequently a contradiction between the polyphonic linear system and the heard result, which is surface or mass."

Under some more freedom in selecting tiling elements, Johnson (2001) has heuristically constructed a simple £nite canon (as opposed to unending canon) and asked for the existence of other solutions. In addition to the basic rhythmic pattern he has used its augmentation (with double durations) like in Bach's *The Art of the Fugue*. However, the available methods were adaptable neither to such a general case, nor to the linear time (as opposed to the circular time).

The given article provides a numerical solution to the general problem. We developed an algorithm for constructing simple canons from several rhythmic patterns, in particular, from successive augmentations of the theme. As for an analytical solution, it was shown that the problem is equivalent to solving Diophantine equations (supposedly 325–409) in special polynomials. For this purpose an isomorphism between rhythmic canons and these polynomials was established. Finally, we describe an application of the method described to algorithmic composition.

In Section 2, "Problem formulation", we introduce basic assumptions and illustrate them with an example.

In Section 3, "Polynomial representation", we introduce an isomorphism between rhythms and 0–1 polynomials, that is, whose coef£cients are 0s and 1s, the same as for representing the structure of sound spectra (Tangian 1993, 1995, 2001). Then the problem of constructing rhythmic canons is reformulated as £nding sums of products of 0–1 polynomials, which is analogous to Diophantine equations in 0–1 polynomials. Since no general solution is known for Diophantine equations already in integers, there is little hope to solve them in polynomials (polynomials generalize integers, containing them as polynomials of degree 0). Respectively, the question of analytically constructing rhythmic canons remains open.

In Section 4, "Algorithm", we introduce a coding convention for rhythmic canons with no redundancy and propose an enumeration algorithm. Its idea is similar to that of the sieve of Eratosthene (284–192 BC) for £nding prime numbers.

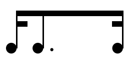
We provide some details on the processing. A sample output of the program is given in Section 8.

In Section 5, "Musical application", we describe the use of the computer output for making a musical piece. Its score is given in Section 9.

In Section 6, "Conclusion" we recapitulate the main results of the paper.

2 Problem formulation

To be speci£c, consider Johnson's (2001) rhythm and its coding by 0s and 1s with respect to a pulse train of sixteenths:



1 1 0 0 1

We are going to build rhythmic canons from this pattern and its augmentations shown in Table 1.

Table 1: Three rhythmic patterns coded by 1s and 0s

Patt. No.	Musical meaning	Progression of tone onsets and empty beats
1	Theme	11001
2	Augmentation	101000001
3	Double augmentation	10001000000000001

To provide a homogeneous pulse train required in rhythmic canons, assume the following:

Assumption 1 (No gap) *Tone onsets result in a regular pulse (= no simultaneous 0s in all the voices).*

Assumption 2 (No double beat) *No tone onset occurs simultaneously in any of two voices (= no simultaneous 1s in any of two voices).*

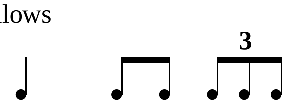
Table 2 depicts the score of a rhythmic canon which satis£es both assumptions.

Table 2: A score of rhythmic canon 11211 with no gaps and no double-beats

Voice number	Pattern number	Beat number														
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	1	0	0	1	.	.	.	.	.	.	.	.	.	.
2	1	.	.	1	1	0	0	1	.	.	.	.	.	.	.	.
3	2	.	.	.	.	.	1	0	1	0	0	0	0	0	1	.
4	1	.	.	.	.	.	.	.	.	1	1	0	0	1	.	.
5	1	.	.	.	.	.	.	.	.	.	.	1	1	0	0	1
Simultaneous onsets		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

The canon code '11211' is the succession of patterns as they enter in the canon given in the second column of Table 2. In the score, 1s are tone onsets, zeros denote sustained tones (tied notes), and points denote sixteenth rests.

Coding a rhythm by a sequence of 0s and 1s is feasible for all notationable rhythms, provided that the reference pulse train is suf£ciently dense, being a common divisor of the durations considered. For instance, a quarter note, two eights, and three eight triplets can be coded as follows



100000100100101010

3 Polynomial representation

De£ne an isomorphism between rhythms and polynomials with coef£cients 0 or 1 as follows. To be speci£c, represent the £rst pattern in Table 1 as follows:

$$P = 11001 \longleftrightarrow p(x) = 1 + 1x + 0x^2 + 0x^3 + 1x^4 \ .$$

If pattern P delays by 2 beats as in the second voice in Table 2, multiply $p(x)$ by x^2:

$$P_2 = 0011001 \quad \longleftrightarrow \quad p(x)x^2 \ .$$

No shift corresponds to the multiplication of $p(x)$ by the polynomial unit 1 .

A superposition of rhythmic patterns corresponds to the sum of the associated polynomials. For instance, the superposition of P and P_2 :

$$P+P_2 = 1111101 \leftrightarrow p(x)+p(x)x^2 = p(x)(1+x^2) \ .$$

A double beat results in a coef£cient 2 instead of 1 for a single beat:

$$P+P_3 = 11012001 \leftrightarrow p(x)+p(x)x^3 = p(x)(1+x^3).$$

Multiple superpositions of $P \leftrightarrow p(x)$ with delays correspond to polynomial products $p(x)q(x)$ with $q(x)$ representing multiple time delays. For instance, the superposition of P with delays by 2, 8, and 10 beats (sum of voices 1, 2, 4, and 5 in Table 2) corresponds to

$$p(x)q(x), \quad \text{where} \quad q(x) = 1 + x^2 + x^8 + x^{10} \ .$$

Let voice delays in a rhythmic canon generated by pattern $P \leftrightarrow p(x)$ be represented by polynomial $q(x)$. Assumptions 1–2 mean that

$$p(x)q(x) = I_n(x) = \sum_{i=0}^{n} x^n \ , \qquad (1)$$

where n is the sum of degrees of $p(x)$ and $q(x)$. In this case, the length of the canon is $n+1$ beats.

Proposition 1 (Uniqueness of a rhythmic canon) *A rhythmic canon generated by pattern $P \leftrightarrow p(x)$ can be $n+1$ beats long if and only if there exists a polynomial $q(x)$ with coef£cients 0 or 1, satisfying condition (1). If such a canon exists, it is unique to within permutation and union of voices.*

The reservation "unique to within permutation and union of voices" in Proposition 1 means that no new canon emerges if we (a) renumber the voices, or (b) reduce the number of voices by putting disjoint rhythmic patterns into the same voice. For instance, £ve voices in Table 2 can be reduced to three voices by uniting the voices 1–3 and 2–5.

Now note that the jth augmentation P_j of pattern P corresponds to the polynomial

$$P_j \quad \longleftrightarrow \quad p_j(x) = p(x^{2^j}) \ .$$

For instance the augmentations from Table 1 correspond to the polynomials

$$\begin{array}{rcl} \text{1st augmentation} & \longleftrightarrow & p(x^2) = 1 + x^2 + x^8 \\ \text{2nd augmentation} & \longleftrightarrow & p(x^4) = 1 + x^4 + x^{16} \ . \end{array}$$

Consequently, a rhythmic canon built from the rhythmical 'theme' P and its two successive augmentations must satisfy the polynomial equation

$$p(x)q(x) + p(x^2)q_1(x) + p(x^4)q_2(x) = I_n(x) \ , \qquad (2)$$

where polynomial $q_j(x)$ is associated with entry delays of the jth augmentation. For example, the canon in Table 2 satis£es the equation (2) for the following polynomials:

$$\begin{aligned} q(x) &= 1 + x^2 + x^8 + x^{10} \\ q_1(x) &= x^5 \\ q_2(x) &= 0 \\ I_n(x) &= 1 + x + \cdots + x^{14} \ . \end{aligned}$$

The isomorphism between rhythms and 0–1 polynomials is useful in analyzing properties of rhythmic canons. In particular, it enables to estimate the dif£culties in £nding a general analytical solution of the problem considered.

Note that polynomials are 'generalized numbers':

- They include numbers as polynomials of degree 0.
- Addition, subtraction, multiplication, and division are de£ned for polynomials similarly to that for numbers.
- The division properties of polynomials are similar due to the unique factorization into irreducible polynomials, which are polynomial analogue of prime numbers.
- The polynomial classes inherit some properties of numbers which are used for their coef£cients: one can consider integer coef£cients, or rational coef£cients, or real coef£cients, etc.

From this standpoint, the equation (2) is a polynomial version of Diophantine equation

$$pq + p_1q_1 + p_2q_2 = I$$

with positive integer coef£cients p, p_1, p_2, I and to be solved in positive integers q, q_1, q_2. For instance, the Diophantine equation

$$5q + 7q_1 = 100 \qquad (3)$$

has two solutions, $(6, 10)$ and $(13, 5)$.

The existence of a general analytical solution (with a formula) to (2) would mean the existence of an analytical solution to much more simple Diophantine equations in integers. Since no solution to Diophantine equations is known, there is little hope to solve more general Diophantine equations for polynomials.

By the way recall that Fermat (1601–1665) has formulated his Great Theorem as a margin note in Diophante's *Arithmetic* as a step towards the unsolvable general case.

4 Algorithm

An appropriate coding convention is often a ladder to success in combinatorics. An enumeration algorithm must operate on as few parameters as possible.

Proposition 2 (Coding) *Under Assumptions 1 and 2, a rhythmic canon coded by a succession of entering rhythmic patterns is unique to within permutation and union of voices.*

Under our coding convention, a canon C is determined by a ternary number

$$C = \{\pi_1 \pi_2 \dots \pi_i\}, \qquad \text{where} \quad \pi_i = 1, 2, 3 \ ,$$

which, being represented by rhythmic patterns or polynomials, satis£es Assumptions 1 and 2 (= equation (2)).

For sorting out inappropriate ternary numbers we use a kind of sieve of Eratosthene. The analogy is twofold:

- If we consider an element (canon) then we delete the branch with its successors, which stems from this element.
- We always start with the £rst remaining element.

The Candidates for canon are to be collected in list C of candidates. The kth candidate is a ternary number $C[k]$, e.g. $C[k] = \{112\}$.

The Selected canons, satisfying Assumptions 1–2, are collected in the list of ternary numbers S. For instance, the £rst selected canon is $S[1] = \{11211\}$.

Creating a new element of list C is appending either 1, 2, or 3 to the currently considered ternary number $C[k]$. The new element can be either rejected, or selected into list S, or farther retained in C as a candidate. In the latter case the new ternary number is appended to the end of list C. Since the ternary number currently processed is no longer needed, it is deleted. Therefore, the element currently processed is always the £rst in list C. Thus C is destroyed from the top, appended from the bottom, and some elements of C are moved to S.

More speci£cally, do the following

0. Initialize list C of candidates with the £rst rhythmic pattern (which must have a gap somewhere), e.g., $C[1] = \{1\}$. Initialize list S of selected canons to be the empty list.

1. Represent the pulse train of the £rst candidate for canon $C[1]$ (as a resulting sequence of 0s and 1s) and £nd the £rst gap (0).

   (a) Append 1, 2, or 3 to $C[1]$. It means that the pattern's P_j £rst beat is put at the £rst 0 of pulse train of C[1].

   (b) If the new canon $\{C[1], \pi_j\}$ has no gaps and no double beats (= has only 1s in the beat representation) then append $\{C[k], \pi_j\}$ to list S.

   (c) If some new canon $\{C[1], \pi_j\}$ has no double beats but has gaps (= has no elements greater than 1 and at least one 0 in the beat representation) then append $\{C[1], \pi_j\}$ to list C.

   (d) Delete the currently considered (£rst) candidate $C[1]$ from list C as unnecessary. Return to the beginning of Item 1.

The list of selected canons has no repeats in the sense that no smaller canon is a part of a larger canon. Indeed, if a canon is accomplished then it is moved from the list of candidates to the selected list, leaving no descendants in list C. In other words, each selected canon is continuous, with the end of a rhythmic pattern in one voice occurring in the middle of a rhythmic pattern of some other voice.

The algorithm cannot miss any of canons, because it is based on generating ternary numbers with all branches. Due to restrictions imposed by Assumptions 1–2 the number of branches retained remains within operational limits, enabling us to perform computations in reasonable time.

The implementation of the algorithm includes several technicalities. For instance, the list C of candidates for canon should be stored and processed by portions, otherwise it may become too long causing runs out of memory or very long disk exchanges. The list C is stored in a series of temporary £les with keeping in memory only the £rst £le (to be destroyed from the top) and the last £le (to be appended from the bottom up to a certain size, after that a new £le should be opened).

The second important point is that for each candidate for canon, its pulse train after the £rst gap should be saved. It prevents from remaking the pulse train while appending a rhythmic pattern to the current candidate for canon.

The program has been written in the MATLAB (= MATrix LABoratory) C++-based computer programming environment for matrix and vector operations. Besides £nding canons the program also selects canons according to several practical criteria, as the number of simultaneous voices, average pattern density, and repeats in the canon structure.

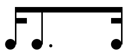
The program output is a LaTeX text £le. A typical processing summary for a program running a PC with a Pentium 300MHz-processor (note that MATLAB is not a compiler but an interpreter) is given in Table 3.

5 Musical application

Section 9 contains the score of *Eine kleine Mathmusic*. It is a G major piece for a woodwind sextet based on a number of rhythmic canons computed. All

Table 3: Processing summary for computing rhythmic canons

Totally tested combinations (candidates for canon)	1260234		
Maximal number of voices in preselection/selection		6	6
Maximal mean pattern number in preselection/selection		1.8	1.7
Periodicity in the preselection/selection		No	Yes
Found/preselected/selected canons of length 5	1	1	0
Found/preselected/selected canons of length 10	6	3	0
Found/preselected/selected canons of length 15	20	0	0
Found/preselected/selected canons of length 20	93	21	1
Found/preselected/selected canons of length 25	348	0	0
Found/preselected/selected canons of length 30	1460	0	0
Found/preselected/selected canons of length 35	5759	0	0
Found/preselected/selected canons of length 40	23502	961	15
Totally found/preselected/selected canons	31189	986	16
Maximal number of £les on disk	120		
Maximum/average number of candidates for canon in memory	1000	296	
Time for computing/selection/making LaTeX £le, in seconds	1856	10	7

the canons are built from the *basic rhythmic pattern* $11001 =$  , its augmentation, and its double augmentation. With regard to the length of the basic pattern, the *time* of the piece is 5/16.

The reverse of the basic code 11001, that is, 10011 determines *basic melodic intervals* which are third and second. Since the piece is assumed tonal, thirds and seconds are not restricted to be minor or major, respectively small or large. In particular, the *theme* motive is g_1, b_1, c_2.

In order to reduce the number of performers, non-overlapping *canon voices* (= entering patterns) are grouped into fewer *physical voices* which are performed by the same instrument. For instance, the canon 11211 shown in Table 2 has £ve canon voices which can be reduced to three physical voices. This is done heuristically with an intention to construct more developed *melodies from successive basic motives*.

Since the piece consists of a series of canons, they are separated by additional 1/16–3/16 rests which are rhythmically perceived as stops and harmonically emphasized as *cadences* at the ends of every canon.

The *style* of the piece is neo-baroque with major-minor harmonies, rules of polyphony, and usual tonal development within a piece. It has a *sonata form* with two themes.

To speak on development, a *variation* convention is accepted. A canon is assumed to be a variation of some other canon if it has the same beginning but a new end, e.g.

$$1121\boxed{1} \quad \rightarrow \quad 1121\boxed{331121}$$

Due to particularities of the algorithm, the list of canons selected is ordered with respect to their size, from shorter to longer, and within every size canons are ordered lexicographically, e.g., canons $112\ldots$ stand before $113\ldots$. That means that closest variations of a given canon succeed it in the list.

The musical form of the piece is displayed in Table 4. As one can see, the harmonic plan of the piece follows classical standards. The £rst entry of the second theme is in the tonality of dominant, the development begins with the £rst theme in the dominant tonality, and the return to the main tonality is performed through the tonality of subdominant. The selection of a particular canon for a particular purpose is motivated by several reasons:

1. For Theme 1, the shortest available canon (No. 1) is selected and used twice with harmonic modi£cation, so that the rhythmic structure of Theme 1 is $1 + 1$.

2. The closest variations of Theme 1, Canons No. 2–4 (the latter taken twice), are used to build a transition to Theme 2. The resulting rhythmic structure of the transition is $2 + 2 + 2 + 2$.

3. Theme 2 (Canon No. 29, the £rst of relative length 4 with fewer than 6 physical voices) is 'slower' due to prevailing patterns of augmentation and second augmentation.

4. Variation of Theme 2 is quite distant (Canon No. 55), but it is the only canon of the same length as Canon No. 29 with only four physical voices. The fewness of physical voices is quite important to preserve harmonic transparency.

5. Development contains the longest canon throughout the piece, with 40 entries of the theme. It has been selected due to its periodicity (which enables to make harmonic sequences usual in classical development) and fewness of physical voices which are 6.

Table 4: The form of *Eine kleine Mathmusik*

Section	Material	Bars	Description
Exposition	Theme 1	1–6	Canon No. 1, twice $\underbrace{11211}_{G\to D}+\underbrace{11211}_{C\to G}$
	Transition 1	7–18	Canons No. 2 and 3 $\underbrace{1121\boxed{331121}}_{C\to C_7}+\underbrace{1121\boxed{332222}}_{F\to F_{6/9}}$
	Transition 2	19–30	Canon No. 4, twice $\underbrace{11\boxed{31211211}}_{d_m\to A_7}+\underbrace{11\boxed{31211211}}_{d_m\to F_6}$
	Theme 2	31–42	Canon No. 29 $\underbrace{112\boxed{22233211131211211}}_{D\to F\sharp_7}$
	Var. Theme 2	43–54	Canon No. 55 $\underbrace{11\boxed{312133112332111211}}_{E\to A_+}$
Development	Theme 1	55–60	Canon No. 1, twice, $D \to A$, $G \to D$
	Var. Trans. 1	61–84	Canon No. 8005 with 3 periods $\underbrace{1}_{G}\underbrace{1222233211}_{D,B_7,a_{m6},E_7}\underbrace{1222233211}_{G,E_7,d_{m6},A_7}\underbrace{1222233211}_{C,A_7,g_{m6},D_7}\underbrace{121332222}_{F\to G_7}$
	Var. Trans. 2	85–96	Canon No. 49 $\underbrace{1131211\boxed{3121131211211}}_{c_m\to A\flat}$
	Theme 2	97–108	Canon No. 29, $C \to E_7$
	Var. Theme 2	109–120	Canon No. 55, $D \to G_+$
Recapitulation	Theme 1	121–126	Canon No. 1, twice, $G \to D$ $C \to G$
	Trans. 2	127–138	Canon No. 4, twice, $g_m \to D_7$, $g_m \to E\flat$
Coda	Theme 1	151–162	Canon No. 1, four times $g_m \to D$, $B\flat \to F$, $f_m \to c_m$, $D_{-9} \to G$

Another selection criterion is the *pattern density*, that is, the mean value of the patterns used in the canon. For instance canon 11211 has the density 6/5, indicating that the basic rhythmic pattern prevails over the augmentations. It implies a better recognizability of the theme and more vivid melodic development. Conversely, for ‘slow’ sections a high density may be desired. In our piece a low density is always preferred.

6 Conclusion

The paper suggests an algorithmic solution to the problem of £nding rhythmic canons with augmentations.

Instead of augmentations of the theme, the model can operate with some other arbitrary rhythmic patterns. Thus besides rhythmic canons restricted to a single theme, one can construct ‘rhythmic fugues’ with several themes and counter-subjects.

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8 Appendix 1: Sample output

Canon No. 1 of length 15 beats with 3 simultaneous voices and pattern density 1.2

V-ce	Patt.	Score		
1	1	11001	.....	.....
2	1	..110	01...	.....
3	2	.....	10100	0001.
4	1	.....	...11	001..
5	1	.....	.....	11001

Canon No. 2 of length 30 beats with 4 simultaneous voices and pattern density 1.6

V-ce	Patt.	Score					
1	1	11001	.....	.....	.....	.....	.....
2	1	..110	01...	.....	.....	.....	.....
3	2	.....	10100	0001.	.....	.....	.....
4	1	.....	...11	001..	.....	.....	.....
5	3	.....	.....	10001	00000	00000	01...
6	3	.....	.....	.1000	10000	00000	001..
7	1	.....	.....	.....	.1100	1....	.....
8	1	.....	.....	.....	...11	001..	.....
9	2	.....	.....	.....	.....	.1010	00001
10	1	.....	.....	.....	.....	1	1001.

Canon No. 3 of length 30 beats with 6 simultaneous voices and pattern density 1.9

V-ce	Patt.	Score					
1	1	11001	.....	.....	.....	.....	.....
2	1	..110	01...	.....	.....	.....	.....
3	2	.....	10100	0001.	.....	.....	.....
4	1	.....	...11	001..	.....	.....	.....
5	3	.....	.....	10001	00000	00000	01...
6	3	.....	.....	.1000	10000	00000	001..
7	2	.....	.....	.....	.1010	00001	.....
8	2	.....	.....	.....	..101	00000	1....
9	2	.....	.....	.....	.....	10100	0001.
10	2	.....	.....	.....	.....	.1010	00001

Canon No. 4 of length 30 beats with 4 simultaneous voices and pattern density 1.4

V-ce	Patt.	Score					
1	1	11001	.....	.....	.....	.....	.....
2	1	..110	01...	.....	.....	.....	.....
3	3	.....	10001	00000	00000	01...	.....
4	1	.....	..110	01...	.....	.....	.....
5	2	.....	.....	10100	0001.	.....	.....
6	1	.....	.....	...11	001..	.....	.....
7	1	.....	.....	.....	11001	.....	.....
8	2	.....	.....	.....	.....	10100	0001.
9	1	.....	.....	.....	.....	...11	001..
10	1	.....	.....	.....	.....	.....	11001

9 Appendix 2: Exposition of *Eine kleine Mathmusic*