Tiling the Line (Pavage de la ligne)
Self-Replicating Melodies, Rhythmic Canons, and an Open Problem

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Summary: Melodic loops can be broken up into modules or tiles that are identical or multiples of one another. Recent investigations, following precedents in some 12-tone theory, and in the work of D. T. Vuza, reveal some new ways of doing this, as well as some new unanswered questions.

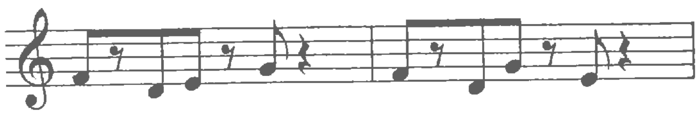
Tiling the plane (pavages en deux dimensions) has been studied ever since the Greeks demonstrated how to do this with regular polygons. Such information was so interesting for geometry, and so useful in architecture, mosaics, and all the "decorative" arts, that regular advancements have been made until today, when one can find large books cataloguing hundreds of tilings or tessellations.

Tiling the line, however, which is just doing the same thing in one dimension, has been rarely treated, and almost exclusively as a musical problem. Probably the first serious studies were about tiling the chromatic scale with combinatorial hexachords and tetrachords and such, for composing 12-tone music. D. T. Vuza, taking the problem into the time dimension, advanced our knowledge beyond the cyclical group Z/12Z and studied "rhythmic canons" of many different lengths, but we still don't know very much, though it is beginning to seem that tiling the line may one day be almost as rich and complex a subject as tiling the plane. And for a minimalist like myself, who has nothing against repeating loops, and who seeks music with very tight logical organization, there is much potential here.

To begin, let me show you the loop used in the *Kientzy Loops*, written for saxophone and tape for Daniel Kientzy. It is an eight-note loop that constitutes a "self-replicating melody," making a copy of itself at 3 : 1 and also at 5 : 1.

Since writing this piece a year ago, I have become interested in loops that not only make copies of themselves, but which also can be broken up into canons that tile the line. So I thought it would be a good exercise for me now to try to tile this eight-point line with two four-point modules. In other words, I want to construct the eight-note loop as a canon of two similar four-note sub-loops.

The loop contains two G's, two F's, two E's and two D's, so it seemed possible that some combination of the four pitches would be complemented by the other four pitches. I tried all the possibilities and found no way in which this loop can be divided into two four-note sub-loops that are exactly the same, but we can work with this combinatorial rhythm, where the two sub-loops are only slightly different:



Now we can have a self-replicating eight-note loop divided into two four-note sub-loops in such a way that one can hear the complete loop and listen to an inexact two-voice canon at the same time.

Since the basic loop makes a copy of itself at 3 : 1 and 5 : 1, the sub-loops do too, so we can divide one of these loops into three loops, each moving three times slower than the basic 1 : 1 tempo of the other sub-loop, and have an inexact four-voice canon, with the voices in two different tempos. Now our tiling has some holes and some simultaneities, but the notes always occur at exactly the right moments within the original eight-note loop.

Or we can split both sub-loops into three voices and have a six-voice canon, with all the voices at one tempo:

Now the line is filled perfectly, with no simultaneities and no holes, and since the basic loop makes a copy of itself at 5 : 1, as well as 3 : 1, we could also transform our loop into an eight-voice inexact canon with three voices moving 3 times slower than the basic loop and five voices moving five times slower, or a 10-voice inexact canon, with all voices moving at 5 : 1. We could even complicate things further with voices moving 9 or 25 times slower than the basic eight-note pulse, but the general idea should already be clear.

When D. T. Vuza did his basic work on "rhythmic canons," he never considered polyrhythmic situations such as these, where the tiles covering the line are moving at different tempos. Thomas Noll, on the other hand, has taken much interest in these possibilities, and has a great facility for finding polyrhythmic solutions. When I asked him, for example, if it was possible to compose a 14-beat loop that would also be a rhythmic canon in which the voices would have tempos of 3 : 5, he responded immediately:

There is only one interesting solution:

The fundamental rhythm is R = {0, 1, 2, 4, 6, 10, 12}, it has several canons with signatures
{{1, 1}, {1, 3}, {1, 5}, {2, 2}, {2, 4}, {2, 6}, {3, 3}, {3, 5}, {4, 4}, {4, 6}, {5, 5}, {6, 6}}

This is the {3,5} canon:

{{0,3,6,12,4,2,8},{5,10,1,11,7,13,9}}

By "signature," he means the tempo ratios of the tiles employed. Here you can see Noll's solution in musical notation. Each of the eight pitchs represents one of the eight voices, three of which move at 3 : 1 and five of which move at 5 : 1:

Z/14Z: R = 10,12,0,1,2,4,6 Sig.=(3,3,3,5,5,5,5,5)

This example was already mentioned in my lecture on *Objets mathématiques trouvés*, as part of the philosophy-mathematics-music seminar at IRCAM in January, a rather long paper, involving numerous esthetic points and many additional examples of loops, along with other findings of Thomas Noll, and some useful mathematical observations by Markus Reineke. That lecture is easy to find on line, so I won't say any more about that, and will continue here with a new problem, which remains completely open. Here is the way I defined it in correspondence with several colleagues earlier this spring:

I want to write a composition using a rhythmic canon with the rhythm (0,1,4), permitting three different tempos in the ratio of 4 : 2 : 1. Just as one must avoid holes and overlaps in traditional two-dimensional pavings, I want to avoid both simultaneities and empty beats in this composition. The problem is that I don't know whether there are a finite number of solutions, in which case it would be nice to somehow use them all, or whether the number of solutions is infinite, in which case I must find some other way to structure the piece.
Regarding this tiling problem as a game, our first move can be either the basic motif (0,1,4), or one of the two augmentations, multiplied by 2 and by 4, as indicated by the letter "a" here. We could also multiply by other numbers, but let's keep the problem on a manageable level for now:

aa00a (0,1,4)
a0a00000a (0,2,8)
a000a00000000000a (0,4,16)

Taking only the first possibility, there are three possible second moves, which I'll notate as b's:

aabba0b
aa0bab00000b
aab0a0b00000000000b
aa0ba00b00000000000b

Taking only the first of these possibilities, there are two possible third moves that fill up the hole, as shown with c's:

aabbacbc00000c
aabbacb000c00000000000c

Taking only the first of these possibilities, there are several possible fourth moves.

aabbacbcdd00dc
aabbacbc000ddc0d
aabbacbcd0d00c00d
aabbacbc0d0d0c000d
aabbacbc00d0dc0000d
aabbacbc0000dcd00000d
aabbacbcd000dc0000000000d
aabbacbc00d00cd00000000000d
aabbacbc000d0c0d00000000000d
aabbacbc0000dc00d00000000000d

Some of these are dead ends, because they leave behind holes that can never be filled with the tiles available, but some of these fourth moves could be continued with several possible fifth and sixth moves, and of course, there are many possible first, second, third and fourth moves that we have not followed up, so there are many possible tilings. But how many?

The situation is not as complex as a game of chess, which offers 20 possible first moves, but there are a lot of possibilities all the same. Simply counting all the possible solutions for a finite loop of 30 or 60 or 75, or some other number divisible by three, could take quite a bit of time.

Marc Chemillier responded that the problem was difficult, and that I should ask D. T. Vuza Vuza responded that the problem is extremely difficult, and that he is no longer working with musical questions. Maybe I will just find some interesting way of tiling some loop having a legth divisible with three and write the piece anyway, but wouldn't it be nicer if I could know that I was using all the possibilities, or that no one will ever be able to use them all?

References

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